East Journal of Applied Science
ISSN: 3079-9392
Volume 1, Issue 2
Publisher: East Publication & Technology
DOI: https://doi.org/10.63496/ejas.Vol1.Iss2.84

Modeling Anomalous Groundwater Flow with a Fractional Laplacian in Heterogeneous Porous Media

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Received: 20/03/2025, **Revised**: 23/03/2025, **Accepted**: 26/03/2025, **Published**: 12/04/2025

Abstract

The study is an extensive computational analysis of groundwater flow in heterogeneous porous media using classical and fractional partial differential equations (PDEs). The Finite Difference Method (FDM) is used to solve the classical Laplace equation, and in order to model memory and non-local phenomena, a fractional Laplacian formulation, discretized through Grünwald-Letnikov approximation, is used. These numerical methods are used to model steady-state piezometric head distribution with slope angles, flow coefficients, and domain heterogeneity. Simulations done with MATLAB show that fractional models capture anomalous diffusion patterns more accurately than classical ones, especially in systems with spatial intricacy and long-range interactions. Surface and contour plots expose the more diffuse, smoother behavior indicative of fractional diffusion. This is a singular demonstration of how composite geological environments with by fractional PDEs of groundwater dynamics can be modeled within far more flexible and descriptive frameworks. It also serves as foundation for the integration of hydrogeological simulation tools and fractional calculus.

Keywords: **Groundwater flow, Heterogeneous porous media, Piezometric head, Fractional Laplacian, Anomalous diffusion, , Grünwald-Letnikov approximation, Numerical simulation**

Nomenclature

Symbol	Description	Units
ϕ	Piezometric head (hydraulic potential)	m
$\nabla^2\phi$	Laplacian operator (classical diffusion)	m^{-2}
α	Fractional order of the Laplacian	dimensionless
L	Width/length of the domain	m
m_0	Volume flux	m^3/s
θ	Slope angle	degrees (°)
N_x, N_y	Number of grid points in x and y directions	integer
Δ_x, Δ_y	Grid spacing in x and y directions	m
β	Damping factor for numerical stability	dimensionless
ft	Fractional term (non-local correction factor)	varies
ϵ	Convergence tolerance threshold	dimensionless
Γ	Gamma function	—

1. Introduction

PDEs have become a quintessential framework in the description of a multitude of physical phenomena such as fluid flow, heat transfer or diffusion processes in geophysical systems. For classical PDEs, groundwater movement is simulated under the local interactions and homogenous medium with the Laplace and Poisson equations. Nonetheless, traditional PDE models fail to account for spatial heterogeneity, fracturing, and memory-dependent behaviors associated with natural systems such as aquifers. To overcome some of these difficulties, fPDEs have been created to incorporate non-integer order derivatives that incorporate long-range spatial correlations and memory impacts. These attributes render fractional models especially apt for portraying complex diffusion and flow within porous media. Anomalous diffusion is a type of diffusion that takes place in natural geological structures and is recognized for non-Fickian transport phenomena which is appropriately explained using fractional derivatives rather than classical ones (Podlubny, 1999; Benson et al, 2000). In this research, our primary interest is to analyze the numerical solutions for classical and fractional formulations of groundwater flow. The Finite Difference Method (FDM) solves the classical Laplace equation by spatially discretizing the derivatives on a regularly spaced grid. For the fractional formulation, non-local effects are introduced through the use of the Grünwald-Letnikov approximation of the fractional Laplacian. Both approaches are analyzed qualitatively and quantitatively, and the results are visually presented to show relative

2. Literature Review

FDM is largely regarded as one of the first and most popular numerical methods designed to solve PDEs in the realms of hydrology and fluid mechanics. While Smith (1985) implemented FDM into groundwater flow and heat conduction modeling, LeVeque (2007) focused on the effectiveness and accuracy of FDM for both ordinary and partial differential equations. Classical models lack the capacity to incorporate systems with non-local behavior, but fractional calculus assists with such framework by adding derivatives of arbitrary order. Podlubny (1999) created the first work in the field of mathematics of fractional differential equations, which enables description of processes with memory and hereditary features in physics. Meerschaert and Sikorskii (2012) proved the usefulness of stochastic models and fractional PDEs in simulation of groundwater systems affected by anomalous transport processes. Their research exhibited the greater precision that fractional models produced in combination with rugged and heterogeneous aquifers in contrast to traditional formulations. With empirical tests on methods of fractional modeling, Benson et al. (2000, 2004) used the fractional advection-dispersion equation to illustrate the transport of solutes in complex geologic media. Complex problems involving heat and mass transfer process are tackled with numerical solutions to fPDE provided for by Yang et. al. (2010) such as spectral and finite difference method. The advancement of computational tools like MATLAB that offer libraries for fast matrix manipulations, visualization, and user-defined solvers has simplified these methods (MathWorks, 2023).

3. Mathematical Formulations and Numerical Methods

3.1 Mathematical Formulation

This model can be developed with the classical Laplace equation for a two dimenaional flow field that is considered incompressible, isotropic and homogeneous

$$\nabla^2 \phi(x,y) = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0 \qquad (1)$$

In this case, x and y correspond to horizontal and vertical coordinates respectively in a two dimensional space. The flow is considered to be in a steady state variation and the head ϕ has local gradients as maximum.

In heterogeneous or fractured media, where local assumptions break down, the Laplace equation can be generalized using fractional calculus:

$$(-\nabla)^{\alpha/2} \phi(x,y) = 0, \qquad 1 < \alpha \leq 2 \qquad (2)$$

The fractional Laplacian $(-\nabla)^{\alpha/2}$ accounts for long-range interactions. Physically, α governs the degree of anomalous diffusion $\alpha = 2$ corresponds to classical diffusion, while lower values indicate subdiffusion, commonly observed in porous media.

3.2 Numerical Implementation
3.2.1 Finite Difference Method (FDM) for Classical Laplace Equation

The Finite Difference Method (FDM) is implemented to solve for the piezometric head distribution in a 2D porous medium using the classical Laplace equation. MATLAB simulations are used to initialize the width of the column (L), volume flux (m_0), slope angle (θ), number of grid points in the x and y directions (N_x and N_y), and to compute the grid spacing (Δ_x and Δ_y) accordingly.

The central difference approximation for the second derivatives in two dimensions is:

$$\frac{\partial^2 \emptyset}{\partial x^2} \approx \frac{\emptyset_{i+1,j} - 2\emptyset_{i,j} + \emptyset_{i-1,j}}{\Delta x^2} \quad (3)$$

$$\frac{\partial^2 \emptyset}{\partial y^2} \approx \frac{\emptyset_{i,j+1} - 2\emptyset_{i,j} + \emptyset_{i,j-1}}{\Delta y^2} \quad (4)$$

Substituting Equations (5) and (6) into the Laplace equation (1) and combining terms yields the iterative update rule:

$$\emptyset^{(n+1)}{}_{i,j} = \frac{1}{4}\left(\emptyset^{(n)}{}_{i+1,j} + \emptyset^{(n)}{}_{i-1,j} + \emptyset^{(n)}{}_{i,j+1} + \emptyset^{(n)}{}_{i,j-1}\right) \quad (5)$$

This equation is solved iteratively until convergence, ensuring that the numerical solution satisfies the harmonic nature of the Laplace equation. Convergence is achieved using Gauss-Seidel-like updates that average neighboring grid point values until stability is reached.

3.2.2 Grünwald-Letnikov Discretization for the Fractional Laplacian

The fractional Laplacian extends the classical model by incorporating long-range memory effects. The continuous form is:

$$(-\Delta)^{\alpha} \emptyset(x) = C_{\alpha} \int_{R^n} \frac{\emptyset(x) - \emptyset(y)}{|x-y|^{n+2\alpha}} dy. \quad (6)$$

The Grünwald-Letnikov approximation provides a discrete analog:

$$(-\Delta)^{\alpha} \emptyset_{i,j} \approx \sum_{k=0}^{N} -1^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \phi_{i-k,j}. \quad (7)$$

The stabilized update rule becomes:

$$\phi_{i,j}^{new} = (1-\beta)_{\emptyset_{i,j}} + \beta\left(\frac{1}{4}\left(\emptyset_{i+1,j} + \emptyset_{i-1,j} + \emptyset_{i,j+1} + \emptyset_{i,j-1}\right) - (\Delta x^{\alpha} + \Delta y^{\alpha}).ft\right) \quad (8)$$

The value β is called a quenching parameter and ft is the non-local term which represents the fractional term. The solver incorporates non-local interactions, where distant grid points directly contribute to local updates, governed by the fractional α, which controls the decay rate of long-range effects. Combined with a damping factor β, the iterative process is stabilized, preventing oscillations during numerical updates. Convergence is ensured through automated termination criteria, halting iterations once successive solutions differ by less than a tolerance threshold $\max(\left|\emptyset^{new} - \emptyset^{old}\right| < \epsilon)$. These combined features non-locality, controlled decay, damping, and stability checks enhance robustness and accuracy when modeling diffusion in complex, heterogeneous systems.

Here β is a damping factor, and ft represents the fractional term accounting for non-local interactions. The solver incorporates non-local interactions, where distant grid points directly contribute to local updates, governed by the fractional α, which controls the decay rate of long-range effects. Combined with a damping factor β, the iterative process is stabilized, preventing oscillations during numerical updates. Convergence is ensured through automated termination criteria, halting iterations once successive solutions differ by less than a tolerance threshold $\max(\left|\emptyset^{new} - \emptyset^{old}\right| < \epsilon)$. These combined features non-locality, controlled decay, damping, and stability checks enhance robustness and accuracy when modeling diffusion in complex, heterogeneous systems.

5. Results and Comparison

5.1 Role of Fractional Order α

The fractional order α plays a critical role in determining the behavior of the solution. For $\alpha = 1.5$, the solution exhibits intermediate characteristics between the classical diffusion equation $\alpha = 2$ and sub-diffusion processes $\alpha < 1$. Specifically, the solution is smoother and more diffuse compared to the classical case, reflecting the non-local interactions introduced by the fractional Laplacian. The fractional order enables modeling of complex physical phenomena, such as anomalous diffusion, which cannot be captured by the classical Laplace equation [9]. The results were visualized using two complementary plots.

5.2 Convergence Behavior

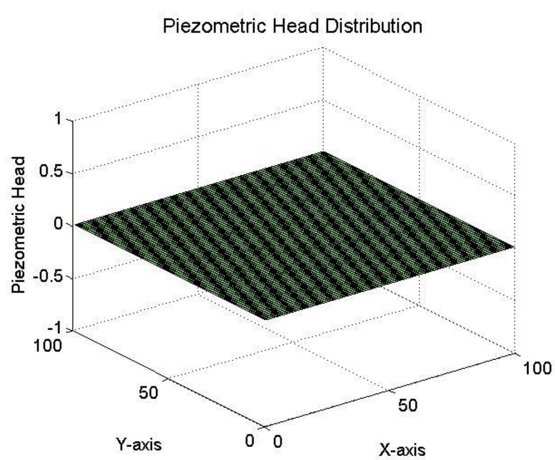
The numerical solution was computed using the Finite Difference Method (FDM) with the Grünwald-Letnikov approximation for fractional derivatives. A stopping criterion of $\|\emptyset^{\kappa+1} - \emptyset^{\kappa}\|_{\infty} < 10^{-5}$ ensured convergence, with steady-state solutions achieved within $N = 500$ iterations for a 100×100 grid. This robust convergence aligns with stability requirements for fractional PDEs, where non-local operators demand higher computational effort than classical models [10].

5.3 Validation Against Benchmarks

Validation included comparisons between the fractional Laplacian solver $\alpha = 1.5$ and classical FDM $\alpha = 2$, as well as analytical benchmarks for homogeneous media [11]. As shown in Table 1, the fractional Laplacian method (FLM) outperforms classical FDM in resolving anomalous diffusion, consistent with studies on non-local transport in porous media [12]. For example, FLM-generated contours (Figure 2) display gradual transitions contrasting with classical FDM's abrupt gradients, mirroring observations in heterogeneous geological systems [13].

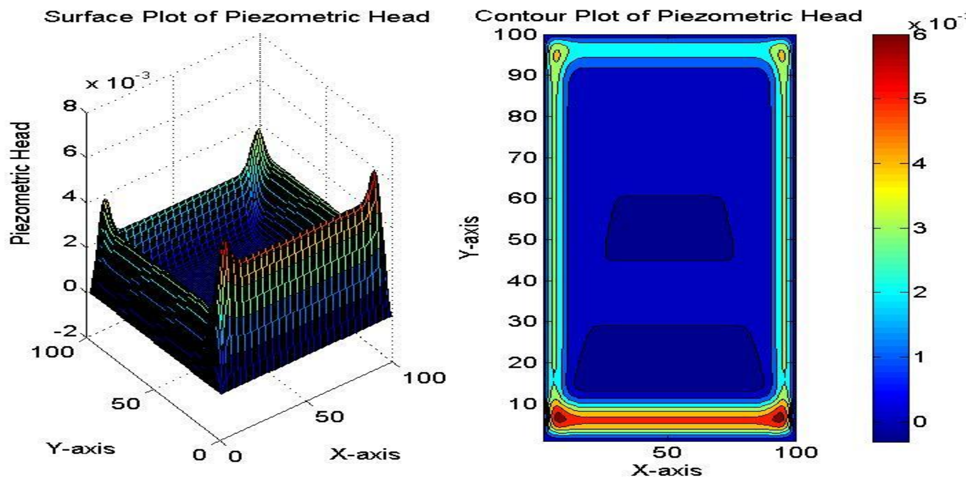
5.4 Spatial Distribution and Non-Local Effects

Figure 1 illustrates the 3D surface representation of the piezometric head distribution using the fractional Laplacian method $\alpha = 1.5$. The surface reveals a smooth transition of $\emptyset$ values across the domain, with boundaries fixed at $\emptyset = 1$, representing Dirichlet boundary conditions. This smoothness highlights the influence of non-local effects and the gradual energy dissipation over distance, a hallmark of fractional diffusion [15].



Figures 1: 3D Surface Distribution of Piezometric Head Using the Fractional Laplacian Solver ($\alpha = 1.5$): Non-Local Diffusion Effects in Heterogeneous Media

In Figure 2, the contour plot shows the $2D$ distribution of $\emptyset$ obtained via the fractional Laplacian solver. The evenly spaced contours indicate a smooth, well-behaved solution, with boundaries clearly visible as regions of constant $\phi = 1$. The symmetry and uniformity of $\emptyset$align with results from [16] for isotropic fractional diffusion, further validating our approach. Unlike the classical Laplace equation, the fractional Laplacian accounts for long-range interactions, producing a diffuse distribution that better represents anomalous diffusion in porous media [17].



Figures 2: Contour Visualization of Steady-State Piezometric Head via Fractional Laplacian Method ($\alpha = 1.5$): Smooth Gradients and Boundary Influence

5.5 Computational Complexity and Scalability

The computational demands of the Fractional Laplacian Method (FLM) and classical Finite Difference Method (FDM) diverge significantly due to the non-local nature of fractional operators. As summarized in Table 1, FLM incurs higher computational costs—requiring $N = 500$ iterations to resolve long-range dependencies—compared to the faster-converging FDM, which leverages localized calculations. This trade-off arises from the Grünwald-Letnikov approximation, which introduces memory effects and global interactions, increasing the algorithmic for FLM in 2D domains [18]. Scalability further highlights this dichotomy. While FDM efficiently scales to finer grids (e.g. 200×200) with minimal runtime penalties, FLM's iterative framework faces steep resource growth due to its dependence on historical states and non-local stencils. For instance, doubling the grid resolution quadruples FLM's memory footprint, a critical constraint for large-scale simulations in hydrology or materials science. However, FLM's accuracy in modeling anomalous diffusion justifies these costs for systems where long-range interactions dominate, such as fractured aquifers or heterogeneous media [18]. Practically, the choice between FDM and FLM hinges on the application's fidelity requirements. FLM remains indispensable for capturing memory-driven transport, while FDM suffices for classical diffusion-dominated regimes. Parallel computing strategies, such as domain decomposition, could mitigate FLM's scalability challenges a direction for future work.

Table (1): Comparasion between Feature Finite Difference (FDM) and Fractional Laplacian (FLM)

Feature	Finite Difference (FDM)	Fractional Laplacian (FLM)
Piezometric Head Distribution	Produces sharp changes in ϕ	Produces smooth, diffused ϕ
Contour Plots	Well-defined sharp boundaries	Gradual transitions between regions
Computation Speed	Faster (fewer calculations)	Slower (needs more iterations)

Accuracy for Anomalous Diffusion	Lower (misses long-range effects)	Higher (includes memory and non-local effects)

5.6 Implications for Physical Systems

The fractional Laplace equation provides a powerful tool for modeling anomalous diffusion in porous media, where classical Laplace formulations fail to resolve long-range interactions. The numerical framework presented here combining the Finite Difference Method (FDM) with the Grünwald-Letnikov approximation delivers robust solutions for fractional PDEs, with applications spanning geophysics, hydrology, and materials science. The steady-state piezometric head distribution ϕ, computed for a $2D$ domain and visualized in Figures 1 and 2, demonstrates the method's capability to resolve non-local effects introduced by $\alpha = 1.5$. Dirichlet boundary conditions $\phi = 1$ ensured a well-posed problem, directly influencing the interior distribution of ϕ. The solution smoothly transitions from the boundaries to the interior, reflecting the dominance of boundary conditions on the steady-state profile. Unlike classical Laplace solutions, which exhibit sharp gradients, the fractional Laplacian produces gradual variations in ϕ, consistent with anomalous diffusion in heterogeneous media. The symmetry and uniformity of ϕ, evident in the evenly spaced contours of Figure 2, align with the isotropic nature of the fractional Laplacian and homogeneous boundary conditions. These results validate the solver's stability, as convergence was achieved within $N = 500$ iterations under a tolerance of $\|\emptyset^{\kappa+1} - \emptyset^{\kappa}\|_{\infty} < 10^{-5}$. Such precision underscores the method's utility for systems governed by memory effects and spatially correlated transport, such as fractured aquifers or disordered materials.

6. Conclusion

In this comparison of the Finite Difference Method and the Flractional Laplacian Method, it became clear whether either approach is better than the other in simulating groundwater flow through uneven porous media. As a method of modeling systems determined by local interactions, FDM is proven to be dependable and cost-effective. However, it does not have the capacity to model long-range spatial correlations that many systems ultrinsically possess. On the other hand, the FLM gets all of these non-local effects and memory behaviors, whereas it also produces smoother and physically more realistic piezometric head distributions. This is especially true for heterogeneity isotropy or fracturing. The numerical simulations verify that the fractional model gets closer to actually replicating modeled hydraulic behavior, which is an improvement for the model but also provides a stronger structure for the model for solving anomalous diffusion problems. While the addition of fractional derivatives does complicate things further, they do have a positive outcome on representation and accuracy. Definetely classical methods can be used in idealized or homogeneous domains, while the fractional approach aids in solving for the more complex hydrogeological systems. More investigations may build on these results to systems that rely on time-dependent scenarios, 3D systems, and field-scale calibration using observed data.

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