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The Role of Nevanlinna and Carleson Functions in Analytic Maps

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***Abstract*:**

We demonstrate that the maximal Nevanlinna counting function and the Carleson function for analytic self-maps of the unit disk are comparable, up to constant factors. This equivalence establishes a fundamental connection between these two classical concepts in complex analysis, providing insights into composition operators and their compactness criteria. The results unify previous findings and highlight the role of Carleson measures and Nevanlinna counting functions in the study of analytic maps.

Keywords: Nevanlinna counting function **,** Carleson function, analytic self-maps, composition operators

1. Introduction

Carleson measures and the Nevanlinna counting function are two classical concepts in Complex Analysis, which are closely related to composition operators in a way which we begin by recalling in this introduction.

According to the well-known Littlewood subordination principle (see [1]), any analytic self-map φ (often called Schur function) of the unit disk D generates abounded composition operator C_φ on the Hardspace H^2(and all Hardy spaces H^p

Now, it follows from the famous Carleson embedding theorem of 1962 [2] that, given a finite positive measure μ on the closed unit disk $\mathbb{D}$ see[3] Jensen formula and the Nevanlinna theory of defect (see [4] or [5]) for meromorphic functions .If the boundedness of $C_\varphi : H^2 \to H^2$ is thus seen to be automatic, its compactness (especially when ϕ is highly non-injective) does not always hold, and is a much more delicate problem. But it turns out that its solution can again be given in terms of Carleson measures or of Nevanlinna counting functions. Indeed, the following little-on theorem was proved by McCluer in 1985 ([6]—see also [7), in terms of so-called vanishing Carleson measures:

$$C_\varphi : H^2 \to H^2 \text{is compact} \Longleftrightarrow p_\varphi(h) = o\,(h)\; as\; h \to 0.$$

Then, in 1987, Shapiro [3] proved that, The usual Littlewood–Paley identity played an essential role in his work. A similar situation occurred for the membership of C_φ in a Schatten class S_p. In 1987, Luecking [8] proved the following Carleson measure type rwhere the Rn, j are certain subsets of D. A bit later, Luecking and Zhu [9] proved that, ψ being related to the Nevanlinna counting function assault to the Nevanlinna counting function

$$C_\varphi : H^2 \to H^2 \in S_p \leftrightarrow \psi \in L^P(\Lambda)$$

where λ is the hyperbolic area measure of $\mathbb{D}$.

Some results of a similar flavor, either in terms of Carleson measures or in terms of Nevanlinna counting functions, can be quoted: for example, Choe [10]and

Theorem (1.1)[11] There exists a universal constant $C > 1$, such that, for every analyticself-map $\phi : D \to D$, one has:

$$\left(\frac{1}{C}\right)\rho_\phi\left(\frac{h}{C}\right) \le \sup_{|w|\ge 1-h} N_\phi(w) \le C\,\rho_\phi(C\,h), \tag{1.1}$$

for $h > 0$ small enough, namely $h \le C^{-1}(1 - |\phi(0)|)$.

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Theorem (1.2)[There exist positive constants C_1, c_1, C_2, c_2, c such that, for every $\xi \in \partial D$, one has, for every $w \in W(\xi, h) \cap D$

(i) $N_\phi(w) \leq C_1 m_\phi[W(\xi, c_1 h)]$;

(ii) $m_\phi[W(\xi, h)] \leq C_2 \frac{1}{h^2} \int_{W(\xi, c_2 h)} N_\phi(w)\, dA(w)$,

for $h > 0$ small enough, namely $h \leq c\,(1 - |\phi(0)|)$.

Our proof gives $C_1 = 24, c_2 = 34, C_1 = 196, and\ C_2 = 128$, and c = 1/16. Nevertheless, these explicit constants are not relevant and we did not try to have "best" constants. It can be shown that for every $\alpha > 1$, there is a constant $C_\alpha > 0$ such that $m_\phi\,(S(\xi, h)) \leq C_\alpha \tilde{v}_\phi(\xi, \alpha h)$ and $\tilde{v}_\phi(\xi, h) \leq C_\alpha\, m_\phi\,(S(\xi, \alpha h))$ for $0 < h < (1 - |\phi(0)|)/\alpha$, where $S(\xi, h)$ is defined in $\tilde{v}(\xi, h) = \min_{w \in S(\xi, h) \cap D} N_\phi(w)$

The result may appear as not so surprising, but it had not been stated, nor proved, within, and helps unifying the theory of composition operators (see [11]). It shows in particular that the results of B. McCluer and J. Shapiro on the one-hand, and of D. Luecking and D. Luecking and K. Zhu on the other hand, are qualitatively the same, even if the quantitative result sharper. This equivalence between Carleson measures and Nevanlinna counting functions also seems of independent interest in the theory of complex variables, and might have other applications. We refer, for instance, to the papers [12,13,14,15]

concerning the Nevanlinna counting function. After this paper was completed we discovered the papers[16,17],whose results differ from ours. The proof was inspired to us by the study [18] of Hardy–Orlicz spaces H attached to a general Orlicz function , which are natural generalizations of Hardy spaces. Although these spaces are not explicitly present in this work, they are lurking behind

the scene. The main tool is a generalization of the Littlewood–Paley identity, under a form due to Stanton (see [19], Theorem 2). This Stanton's formula had been used by Shapiro [3], and later by Choa and Kim [20], but actually only in the Hil bertian case, which gives nothing else than the Littlewood–Paley identity. As pointed out to us by the referee, the Stanton formula was also used in non-Hilbertian cases by Shapiro and Sundberg in [21] and by Liu et al. in [22]. Here, we use the full version of Stanton's formula, and, though our proof is not technically difficult, it is by no way straightforward. The paper is organized as follows. definitions and notations, and in particular attaches to each Schur function φ two maximal functions ρ_φ and v_φ respectively attached to the Carleson measure mϕ and to the Nevanlinna counting function N_φ. Section 2 shows, in a precised sense, that the maximal Nevanlinna counting function v_φ is dominated by the maximal Carleson functionρ_φ ,shows, similarly, that the maximal Carleson function ρ_φ is dominated by the maximal Nevanlinna counting function v_φ.

Lemma (1.3) Let ϕ be an analytic self map of D. For every $z \in D$, one has, if $w = \phi(z), \xi = w/|w|$ and $h = 1 - |w| \leq 1/4$:

$$m_\phi\,(W(\xi, 12\,h)) \geq m_\phi\,(S(\xi, 6h)) \geq \frac{|w|}{8}(1 - |z|) \geq \frac{1}{16}(1 - |z|).$$

Proof We may assume, by making a rotation, that w is real and positive: $3/4 \leq w < 1$.

Let:

$$T\,(u) = \frac{au + 1}{u + a} \qquad (1.2)$$

Where $a = w - \frac{2}{w} < -1$,

so that $T : D \to D$ is analytic, and $T\,(w) = w/2$.

If P_z is the Poisson kernel at z, one has:

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$$\frac{w}{2} = T\,[\phi(z)] = \int_{T} (T \circ \phi)^{*} p_z dm = \int_{T} \Re e[(T \circ \phi)^{*}] p_z dm$$

Hence, if one sets:

$$E = \{\Re e\,(T \circ \phi) \geq w/4\} = \{\Re e\,[(T \circ \phi)^{*}] \geq w/4\},$$

One has

$$\frac{w}{2} = \int_{E} p_z dm + \frac{w}{4}\int_{E^C} p_z dm \leq \int_{E} p_z dm + \frac{w}{4}\int_{T} p_z dm = \int_{E} p_z dm + \frac{w}{4}$$

Therefore

$$\int_{E} p_z dm \geq \frac{w}{4}$$

Since $\|Pz\|_{\infty} \leq \frac{2}{1-|z|}$, we get:

$$m(E) \geq \frac{w}{8}(1 - |z|). \tag{1.3}$$

On the other hand, (25) writes

$$u = T^{-1}(U) = \frac{aU - 1}{a - U} \tag{1.4}$$

$$|1 - z| = |a + 1|\frac{|1 - U|}{|a - U|} \leq \frac{2|a + 1|}{|a - U|}$$

But $a < -1$ is negative, so $\Re eU \geq w/4$ implies that

$$|a - U| \geq \Re e\,(U - a) \geq \frac{w}{4} - a = \frac{2}{w} - \frac{3}{4}\; w \geq \frac{5}{4}$$

Moreover, for $w \geq 3/4$:

$$|a + 1| = (1 - w)\left(\frac{2}{w} + 1\right) \leq \frac{11}{3}(1 - w).$$

We get hence $|1 - u| \leq 6\,h$ when (5-30) holds and $\Re eU \geq w/4$.

It follows that

$$\phi^{*}(E) \subseteq T^{-1}\left(\{\Re eU \geq w/4\}\right) \subseteq S(1, 6h),$$

giving $m_{\phi}\left(W(1, 12h)\right) \geq m_{\phi}\left(S(1, 6h)\right) \geq m(E)$.

Combining this with (26), that finishes the proof.

Theorem (1.4) For every analytic self-map $\phi: D \to D$ and every subharmonic function $G: D \to R$, one has

$$\lim_{r \uparrow 1} \int_{\partial D} G[\phi(r\xi)]dm(\xi) = G[\phi(0)]\frac{1}{2}\int_{D} \Delta G(w) N_{\phi}(w) dA(w) \tag{1.5}$$

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where Δ is the distributional Laplacia

2- Nevanlinna counting function by the Carleson function

[23,24,25,26] concerning the Nevanlinna counting function.

We discovered the papers [9,6], whose results differ from ours. The proof was inspired to us by the study [5] of Hardy–Orlicz spaces attached o a general Orlicz function, which are natural generalizations of Hardy spaces.. The main tool is a generalization of the Littlewood–Paley identity (22), under a form due to Stanton (see [19], Theorem 2). This Stanton's formula had been used by Shapiro [3], and later by Choa and Kim [20], [21] in [22]. but actually only in the Hilbertian case, which gives nothing else than the Littlewood–Paley identity

Theorem (2.1) For every analytic self-map ϕ of D, one has, for every $a \in D$:

$$N_\phi(a) \leq 196 m_\phi\left(W(\xi, 12h)\right), \tag{2.1}$$

for $0 < h < \frac{1-|\phi(0)|}{4}$, where $\xi = \frac{a}{|a|}$ and $h = 1 - |a|$.

In particular, for $0 < h < (1 - |\phi(0)|)/4$:

$$v_\phi(h) = \sup_{|a|\geq 1-h} N_\phi(a) \leq 196\, \rho_\phi(12h). \tag{2.2}$$

Let us note that, since $W(\zeta, s) \subseteq W(\xi, 2t)$ whenever $0 < s \leq t$ and $\zeta \in W(\xi, t) \cap \partial D$, we get from (25) that

$$\sup_{w \in W(\xi,h) \cap D} N_\phi(a) \leq 196 m_\phi\,(W(\xi, 24h))$$

Proof : If $a \notin \phi(D)$, one has $N_\phi(a) = 0$, and the result istrivial. We shall hence assume that $a \in \phi(D)$.

Let ϕ: $[0,\infty) \to [0,\infty)$ be an Orlicz function, that is a non-decreasing convex function such that $\phi(0) = 0$ and $\phi(\infty) = \infty$, and we assume that $\acute{\phi}$ is also an Orlicz function. In other words, $\acute{\acute{\phi}}$ is an arbitrary non-negative and non-decreasing function and $\acute{\phi}(x) = \int_0^x \acute{\acute{\phi}}(t)dt$ and $\phi(x) = \int_0^x \acute{\phi}(t)dt.$

Let now $f : D \to C$ be an analytic function. We have, outside the zeroes of f writing $\Delta\phi(|f|) = 4\partial\,\bar{\partial}\phi(\sqrt{|f|^2})$:

$$\Delta\phi(|f|) = \left[\acute{\acute{\phi}}(|f|) + \frac{\acute{\phi}(|f|)}{|f|}\right]|\acute{f}|^2 \tag{2.3}$$

We shall

$$\Delta\phi(|f|) \geq \ddot{\phi}(|f|)\left|\acute{f}\right|^2 \tag{2.4}$$

(this is a not too crude estimate, since, $\acute{\phi}$ being an Orlicz function $\acute{\acute{\phi}}$ is non-negative and non-decreasing, and hence

$$\acute{\phi}(x) = \int_0^x \acute{\acute{\phi}}(t)dt \leq x\acute{\phi}(x) \text{ and } \int_0^x \acute{\acute{\phi}}(t)dt \geq \int_{x/2}^x \acute{\acute{\phi}}(t)dt \geq (x/2)\acute{\acute{\phi}}(x/2)$$

Set now, for $a \in D$:

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$$f_a(z) = \frac{1-|a|}{1-\bar{a}z} \qquad a \in \bar{D} \tag{2.5}$$

Since $\phi(|f_a|)$ is subharmonic (ϕ being convex and non-decreasing) and bounded, we can use Stanton's formula as:

$$\int_{\partial D} \phi(|f_a \circ \varphi|)|) \, dm \geq \frac{1}{2}\int_D \acute{\acute{\phi}}(|f_a||\acute{f_a}|^2) N_\varphi dA$$

Let $h = 1 - |a|. For\ |z - a| < h$, one has

$|1-\bar{a}z| = |(1-|a|^2) + \bar{a}(a - z)| \leq (1-|a|^2) + |a - z| \leq 3h;$

Hence $|f_a(z)| \geq 1/3$ for $|z - a| < h$. It follows, since $\acute{\acute{\phi}}$ is non-decreasing:

$$\int_{\partial D} \phi(|f_a \circ \varphi|)|) \, dm \geq \frac{1}{2}\acute{\acute{\phi}}\left(\frac{1}{3}\right) \int_{D(a,h)} |\acute{f_a}|^2 N_\phi dA$$

Now, if $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, one has $|f_a(z)| = \frac{|a|}{1+|z|}|\acute{\phi}_a(z)| \geq \frac{3}{7}|\acute{\phi}_a(z)|$ (we may, and do, assume that $1 - |a| = h \leq 1/4$); hence:

$$\int_{\partial D} \phi(|f_a \circ \varphi|)|) \, dm \geq \frac{1}{2}\acute{\acute{\phi}}\left(\frac{1}{3}\right)\frac{9}{49} \int_{D(a,h)} |\acute{\phi}_a|^2 N_\varphi dA$$

$$\frac{9}{98}\acute{\acute{\phi}}\left(\frac{1}{3}\right) \int_{\phi_a(D(a,h))} N_{\phi\circ\varphi} dA$$

because $N_{\phi\circ\varphi}(\phi_a(w)) = N_\varphi(w)$ *and* $\phi_a^{-1} = \varphi_a$

But $\varphi_a(D(a,h)) \supseteq D(0,1/3)$: indeed, if $|w| < 1/3$, then $w = \varphi_a(z)$, with

$$|a - z| = \left|\frac{(1-|a|^2)w}{1-\bar{a}w}\right| \leq (1-|a|^2)\frac{|w|}{1-|w|} < 2h\frac{1/3}{1-1/3} = h$$

We are going now to use the sub-averaging property of the Nevanlinna function ([1 p. 190], [3 Sect. 4.6], or [23Proposition 10.2.4]): for every analytic self-map $\psi : D \to D$, one has

$$N_\phi(w_0) \leq \frac{1}{A(\Delta)}\int_\Delta N_\varphi(w) dA(w) \tag{2.6}$$

for every disk Δ of center w_0 which does not contain $\psi(0)$.

Lemma (2.6) For $1 - |a| < (1 - |\phi(0)|)/4$, one has $|(\phi_a \circ \phi)(0)| > 1/3$.

Lemma (2.7) For every $\xi \in \partial D$ and every $h \in (0, 1/2]$, one has:

$$|1-\bar{a}z|^2 \geq \frac{1}{4}(h^2 + |z - \xi|^2), \qquad \forall z \in \bar{D}, \tag{2.7}$$

where $a = (1 - h)\xi$.

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Theorem (2.8) For every analytic self-map $\phi : D \to D$, one has, for every $\xi \in \partial D$

$$m_\phi(w(\xi,h) \le 64 \sup_{w \in W(\xi,64h) \cap D)} N_\phi(w)$$

for $0 < h < (1 - |\phi(0)|)/16$.

$$v_\phi(\xi,h) \le \sup_{w \in W(\xi,h) \cap D)} N_\phi(w) \tag{2.8}$$

Proof We shall set:

$$v_\phi(\xi,h) = \sup_{w \in W(\xi,h) \cap D)} N_\phi(w)$$

Note that

$$v_\varphi(h) = \sup_{|\xi|=1} v_\phi(\xi,h)$$

where $v_\phi$is defined in

$$v_\varphi(t) = \sup_{|w| \ge 1-t} N_\phi(w)$$

If for some $h_0 > 0$, one has $v_\phi(\xi,h_0) = 0$, then $\phi(D) \subseteq D \setminus W(\xi,h_0)$, and hence $m_\phi(w(\xi,h) = 0$ for $0 < h < h_0$. Therefore we shall assume that $v_\phi(\xi,h) > 0$. We may, and do, also assume that $h \le 1/4$. By replacing ϕ by $e^{i\theta}\phi$, it suffices to estimate $m_\phi(S(1,h))$ (recall that

$S(1,t) = \{z \in D; |1 - z| \le t\}$).

We shall use the same functions f_a as in the proof of Theorem (5.2.1), but, for convenience, with a different notation. We set, for $0 < r < 1$:

$$u(z) = \frac{1-r}{1-rz} \tag{2.9}$$

Let us take an Orlicz function ϕ as in the beginning of the proof of Theorem (2), which will be precised later. We shall take this function in such a way that $\Delta\phi(|u(\varphi(0))|) = 0$.

Since $\acute{\phi}(x) \le x\, \acute{\acute{\phi}}(x)$ (2.3) becomes:

$$\Delta\phi(|u|) \le 2\acute{\acute{\phi}}(|u|)|\acute{u}|^2$$

and Stanton's formula writes, since $\phi(|u(\phi(0))|) = 0$:

$$\int_{\partial D} \phi(|u \circ \varphi|)\, dm \le \int_{D} \acute{\acute{\phi}}\left((|u(z)|)|\acute{u}(z)|^2\right) N_\phi\, dA(z) \tag{2.10}$$

In all the sequel, we shall fix h, $0 < h \le 1/4$, and take $r = 1 - h$.

(i)For $|z| \le 1$ and $|1 - z| \le h$, one has $|1 - rz| = |(1 - z) + hz| \le 2h$, so:

$$|u(z)| \ge \frac{(1-r)}{2h} = \frac{1}{2}$$

$$m_\phi\left(S(1,h)\right) \leq \frac{1}{\phi(1/2)} \int_{S(1,h)} \phi(|u(z)|) dm_\phi(z) \leq \frac{1}{\phi(1/2)} \int_{\bar{D}} \phi(|u(z)|) dm_\phi(z)$$

$$\geq \frac{1}{\phi(1/2)} \int_{T} \phi(|(u \circ \varphi)(z)|) dm(z)$$

and so, by (39):

$$m_\phi\left(S(1,h)\right) \leq \frac{1}{\phi\left(\frac{1}{2}\right)} \int_{\mathrm{D}} \acute{\phi}\left((|u(z)|)|\acute{u}(z)|^2\right) N_\varphi(z) dA(z) \tag{2.11}$$

We are going to estimate this integral by separating two cases: $|1 - z| \leq h$ and $|1 - z| > h$. For convenience, we shall set:

$$\tilde{v}(t) = \sup_{w \in W(1,t) \cap D} N_\phi(w) \tag{2.12}$$

Remark first that

$$|\acute{u}(z)| \leq \frac{h}{(1-r)^2} = \frac{1}{h}$$

Since $|u(z)| \leq 1$, we get hence:

$$\int_{|1-z| \leq h} \acute{\phi}\left((|u(z)|)|\acute{u}(z)|^2\right) N_\phi(z) dA(z) \leq \int_{\mathrm{S(1,h)}} \acute{\phi}(1) \frac{1}{h^2} \tilde{v}(h) dA(z)$$

giving, since $A\left(S(1,h)\right) \leq h^2$:

$$\int_{|1-z| \leq h} \acute{\phi}\left((|u(z)|)|\acute{u}(z)|^2\right) N_\phi(z) dA(z) \leq \acute{\phi}(1)\tilde{v}(h) \tag{2.3}$$

(ii) For $0 < h \leq 1/4$, one has:

$$|u(z)| \leq \frac{2h}{|1-z|} \quad and \ |\acute{u}(z)| \leq \frac{2h}{|1-z|^2}$$

indeed, we have $|1 - rz| = r\left|\frac{1}{2} - z\right| \geq r\,|1 - z|$ (this is obvious, by drawing a picture), and hence$|1 - rz| \geq \frac{3}{4}|1 - z|$,since $r = 1 - h \geq 3/4$. We obtain:

$$\int_{|1-z|>h} \acute{\phi}\,(|u(z)|)|\acute{u}(z)|^2 N_\phi(z) dA(z) \leq 4 \int_{|1-z|>h} \acute{\phi}\,\left(\frac{2\mathrm{h}}{|1-\mathrm{z}|}\right) \frac{\mathrm{h}^2}{|1-\mathrm{z}|^4} \mathrm{N}_\phi(\mathrm{z}) \mathrm{dA}(\mathrm{z})$$

Then, using polar coordinates centered at 1 (note that we only have to integrate over an arc of length less than π), and the obvious inequality $N_\phi(z) \leq \tilde{v}(|1 - z|)$, we get:

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$$\int_{|1-z|\leq h} \acute{\acute{\phi}}\left(|u(z)|\right)|\acute{u}(z)|^2 N_{\phi}(z)dA(z) \leq 4\int_{h} \acute{\acute{\phi}}\left(\frac{2h}{t}\right)\frac{h^2}{t^3}\tilde{v}(t)dt \tag{2.14}$$

We now choose the Orlicz function as follows (with $a = \phi(0)$):

$$\acute{\acute{\phi}}(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq h/(1-|a|), \\ \dfrac{1}{\tilde{v}(2h/v)} & \text{if } h/(1-|a|) < v < 2, \\ \dfrac{1}{\tilde{v}(h)} & \text{if } v \geq 2 \end{cases} \tag{2.15}$$

This function is non-negative and non-decreasing. Moreover, one has $\phi(x) = 0$ for $0 \leq x \leq h/(1-|a|)$. Hence, since $|u(a)| \leq \frac{h}{1-|a|}$, one has $\phi(|u(a)|) = 0$.

Then

$$\int_{h}^{2} \acute{\acute{\phi}}\left(\frac{2h}{t}\right)\frac{h^2}{t^3}\tilde{v}(t)dt = \int_{h}^{2(1-|a|)} \acute{\acute{\phi}}\left(\frac{2h}{t}\right)\frac{h^2}{t^3}\tilde{v}(t)dt \leq \int_{h}^{2}\frac{h^2}{t^3}dt = \frac{1}{2} \tag{2.16}$$

$$\phi\left(\frac{1}{2}\right) = \int_{0}^{1/2} \acute{\phi}(t)\,dt \geq \int_{1/4}^{1/2} \acute{\phi}(t)\,dt \geq \int_{1/4}^{1/2} \frac{t}{2}\acute{\acute{\phi}}\left(\frac{t}{2}\right)dt \geq \frac{3}{64}\acute{\acute{\phi}}\left(\frac{1}{8}\right)$$

When $h < (1-|a|)/8$, one has $1/8 > h/(1-|a|)$; hence $\acute{\acute{\phi}}(1/8) = 1/\tilde{v}(16h)$, and $\acute{\acute{\phi}}(1) = 1/\tilde{v}(2h)$. We get hence, from (40), (42), (43) and (45)

$$m_{\phi}\left(S(1,h)\right) \leq \frac{64}{3}\tilde{v}(16h)\left[\frac{\tilde{v}(h)}{\tilde{v}(2h)} + 2\right] \leq 64\tilde{v}(16h)$$

Since

$W(1,t) \subseteq S(1,2t)$we get $m_{\phi}\left(W(1,h)\right) \leq 64 \min\limits_{w\in S(1,32h)} N_{\varphi}(w)$ for $0 < h < (1\,|\phi(0)|)/16$, and that ends the proof of Theorem (5.2.8), since $S(1,32h) \subseteq W(1,64h)$.

Theorem(2.9) There are universal constants $C, c > 1$ such that

$$m_{\varphi}\left(S(\xi,h)\right) \leq \frac{1}{A\left(S(\xi,ch)\right)}\int_{(S(\xi,c\,h))} N_{\varphi}(z)dA(z)$$

for every analytic self-map $\varphi : D \to D$, every $\xi \in \partial D$, and $0 < h < (1 - |\phi(0)|)/8$.

Proof We are going to follow the proof of Theorem (5.2.8) We shall assume that $\xi = 1$ and we set:

$$I(t) = \int_{S(1,t)} N_{\phi}(z)dA(z) \tag{2.17}$$

Then

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When $|1 - z| < h$, we have, instead of (2.17):

$$\int_{|1-z|\leq h} \acute{\phi}(|u(z)|)|\acute{u}(z)|^2 N_\phi(z)dA(z)$$

$$\leq \int_{S(1,t)} \acute{\phi}(1)\frac{1}{h^2}N_\phi(z)dA(z) = \acute{\phi}(1)\frac{1}{h^2}I(h) \tag{2.18}$$

For $|z - 1| \geq h$, we write:

$$\int_{|1-z|>h} \acute{\phi}(|u(z)|)|\acute{u}(z)|^2 N_\phi(z)dA(z) = \sum_{k=1}^{\infty}\int_{hh\leq|1-z|<(k+1)h} \acute{\phi}(|u(z)|)|\acute{u}(z)|^2 N_\phi(z)dA(z)$$

$$\leq 4\sum_{k=1}^{\infty}\acute{\phi}\left(\frac{2h}{kh}\right)\frac{h^2}{k^4h^4}I\big((k+1)h\big) = 4\sum_{k=1}^{\infty}\acute{\phi}\left(\frac{2}{k}\right)\frac{1}{k^4h^2}I\big((k+1)h\big)$$

We take, with $a = \phi(0)$:

$$\acute{\phi}(v) = \begin{cases} 0 \quad \text{if} \quad 0 \leq v \leq h/(1-|a|)\,, \\ \dfrac{1}{I\left(\left(\frac{2}{v}+I\right)h\right)} \quad \text{if } v > h/(1-|a|)\,. \end{cases} \tag{2.19}$$

$$\int_{|1-z|\geq h} \acute{\phi}(|u(z)|)|\acute{u}(z)|^2 N_\phi(z)dA(z) \leq \frac{4}{h^2}\sum_{k=1}^{\infty}\frac{1}{k^4} \leq \frac{5}{h^2}$$

Since $h < (1 - |a|)/8$, one has $1/8 > h/(1 - |a|)$; hence $\acute{\phi}(1/8) = \frac{1}{I\,(17h)}$ and $\acute{\phi}(1) = \frac{1}{I\,(3h)}$. Therefore

$$m_\phi\ \big(S(\xi,h)\big) \leq \frac{64}{3}I(17h)\left[\frac{1}{h^2}\frac{I(h)}{I(3h)} + \frac{5}{h^2}\right] \leq \frac{64}{3}I(17h)\frac{6}{h^2} \leq 128\times 17^2\frac{\mathrm{I}\,(17\mathrm{h})}{\mathrm{A}(\mathrm{S}\,(1{,}17\mathrm{h}))}$$

Theorem (2.10) There exist a universal constant $K > 0$ such that, for every analytic self-map ϕ of D, one has, for $0 < \varepsilon < 1$:

$$\nu_\varphi(\varepsilon\, t) \leq K\,\varepsilon\,\nu_\varphi(t), \tag{2.20}$$

for t small enough.

$$\nu_\varphi(\xi,\varepsilon t) \leq K\,\varepsilon\,\nu_\varphi(\xi,t), \tag{2.21}$$

where $\nu_\phi(\xi,\varepsilon t) = \sup_{w\in W(\xi,s)\cap D} N_\phi(w)$.

Theorem (2.11) [Let $\varphi : D \to D$ be an analytic self-map and ψ be an Orlicz function. Then the composition operator $C_\phi : H^\Phi \to H^\Psi$ is compact if and only if

$$\sup_{|w|\geq 1-h} N_\phi(w) = O\left(\frac{1}{\psi(A\psi^{-1}(1/h))}\right) \qquad as\quad h\to 0 \quad \forall A>0$$

It should be noted, due to the arbitrary $A > 0$, that (5-51) may be replaced by

$$\sup_{|w|\geq 1-h} N_\phi(w) \leq \frac{1}{\psi(A\psi^{-1}(1/h)} \qquad \forall A > 0$$

for $h \leq h_A$, this condition also writes, setting $\nu_\phi(h) = \sup_{|w|\geq 1-h} N_\phi(w)$ (see (5-50)):

$$\lim_{h\to 0} \frac{\psi^{-1}(1/h)}{\psi^{-1}(1/\nu_\phi(h))} = 0 \tag{2.22}$$

Theorem (2.12) Let $\psi : D \to D$ be an analytic self-map, and ψ be an Orlicz function. Assume that the composition operator $C_\phi : H^\psi \to H^\psi$ is compact. Then:

$$\lim_{|z|\to 0} \frac{\psi^{-1}\left(\frac{1}{1-|z|}\right)}{\psi^{-1}\left(\frac{1}{1-|\phi(z)|}\right)} = \infty \tag{2.23}$$

Conversely, if ϕ is finitely-valent, then (53) suffices for $C_\phi : H^\psi \to H^\psi$ to be compact.

Proof To get the necessity, we could use Theorem (2.11) and the fact that $1-|z| \leq log \frac{1}{|z|} \leq N_\phi\,(\phi(z))$; but we shall give a more elementary proof. Let HM^ψ be the closure of H^∞ in M^ψ. Since $C_\phi(H^\infty) \subseteq H^\infty, C_\phi$ mapsHM^ψ into itself and $C_\phi : M^\psi \to M^\psi$ being compact, its restriction $C_\phi : HM^\psi \to HM^\psi$ is compact too. We know that the evaluation $\delta_a : f \in HM^\psi \to f(a) \in C$ Chas norm$\approx \psi^{-1}\left(\frac{1}{1-|a|}\right)$ [19,Lemma 3.11](see also [262]theorm4.2); hence $\delta_a \,\|\delta_a\| \xrightarrow[|z|\to 1]{} 0$ weak-star (because $|\delta_a(f)| = |\,)| = |f(a)| \leq \|f\|_\infty\, for\, f \in H^\infty).If\;\; C_\phi$ is compact, its adjoint C_ϕ^* is compact as well; we get hence $\left\|C_\phi^*(\delta_a/\|\delta_a\|)\right\| \xrightarrow[|a|\to 1]{} 0$. But $C_\phi^*\delta_a = \delta_{\phi(a)}$. Therefore

$$\frac{\psi^{-1}\left(\frac{1}{1-|\psi(a)|}\right)}{\psi^{-1}\left(\frac{1}{1-|a|}\right)} \xrightarrow[|z|\to 1]{} 0$$

Conversely, assume that (53) holds. For every $A > 0$, one has, for $|z|$ close enough to $\psi^{-1}\left(\frac{1}{1-|z|}\right) \geq A\psi^{-1}\left(\frac{1}{1-|\phi(z)|}\right)$ in other words, one has: $1/\psi A(\psi^{-1}(1/1-|\phi(z)|) \geq 1-|z|$. But, when ϕ is p-valent, and if $w = \phi(z)\, with\, |z| > 0$ minimal, one has $N_\phi(w) \leq p\, log \frac{1}{|z|} \approx 1-|z|$. Since $|z| \to 1$ when $|w| = |\phi(z)| \to 1$ (otherwise, we should have a sequence (z_n) converging to some $z_0 \in D$ and $\phi(z_n)$ would converge to $\phi(z_0) \in D)$, we get $\sup_{w|\geq 1-h} |\, N_\phi(w) \lesssim 1/\psi A\psi^{-1}(1/1-|w|) \leq 1/\psi A\psi^{-1}(1/1-h)$, for h small enough. By Theorem (2.10), with (2.12), that means that C_ϕ is compact on H^ψ

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